Estimating Partisan Bias in Practice

*This an initial draft for review without most citations.*

Abstract

Ten popular measures of partisan bias are evaluated for a cross section of nearly three dozen past, present, and hypothetical congressional plans. Just two, (dis)proportionality and the efficiency gap, are shown to be reliable indicators of partisan bias across a wide range of statewide vote shares. A rule is developed for interpreting raw efficiency gap measurements in context. The proportionality metric is extended to be more robust.

Introduction

This paper proceeds as follows: The first section formally defines bias in practical terms. Section two introduces and categorizes ten metrics: declination (), lopsided outcomes (), mean–median (), seats bias (), votes bias (), geometric seats bias (), global symmetry (), proportional (), efficiency gap (), and gamma (). It predicts how they will perform as measures of partisan bias. Sections three through five apply the metrics to the 2011 congressional plans for 11 states using a composite of 2012 election data, the corresponding 2020 plans for those states using a composite of 2016–2020 election results, and a dozen carefully curated hypothetical plans, respectively. This reveals that , , and don’t measure partisan bias directly, but instead a technique of partisan gerrymandering and that , , , and aren’t reliable indicators of bias, especially in states that are unbalanced politically. is shown to be too permissive to be useful. In section six, the robustness of is increased in two ways.

1. Definition of Partisan Bias

In contrast to more sophisticated notions motivated by legal theory and political science, we propose a *operational* definition of partisan bias for single-member district redistricting plans:

*Partisan bias is the difference between the expected and likely seat shares.*

In our political system dominated by two parties, the number of seats candidates for each party wins determines the control of Congress and state legislatures. Seats won are political currency.

Because district boundaries determine how a state’s votes will likely get translated into seats, redistricting is, in effect, the process of deciding how many seats each party will win.

When state legislators, redistricting commissioners, and political operatives gauge whether a proposed or adopted plan will be fair politically or will favor one party over the other, they compare how votes will likely get translated into seats under the plan to some normative ideal for how they should get translated into seats in a representative democracy. If a plan will likely translate votes into a number of seats that closely matches their ideal seats–votes relationship, they will judge the plan to be politically fair. The more the map will likely translate votes into seats in a way that significantly deviates from that ideal relationship, the more they will judge the plan to be biased in favor of one party or the other.

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To formalize this, call the ideal or expected seats–votes relationship :

(1)

where and are the two-party Democratic vote share and seat share, respectively.[[1]](#footnote-1) As you will see in the section 2, there are many candidates for this expected seats-votes function.

Similarly, call the likely actual seats–votes relationship :

(2)

You can infer the likely seats–votes curve from a typical statewide vote share and the district-by-district vote shares, using a composite of prior elections as a proxy for the not-yet-held election.[[2]](#footnote-2)

In practical terms, bias is simply the difference between these two functions:

(3)

where is the likely statewide vote share.[[3]](#footnote-3) In other words, bias is the difference between the share of seats that *should* be won () and the share of seats *likely* to be won ().

We can infer and stipulate three additional requirements for valid measures of bias.

First, since a likely seats–votes curve maps vote shares onto seat shares, any valid measure of bias must be a function that take two parameters – a likely statewide vote share and the likely corresponding seat share – and returns a seat share difference:

(4)

Second, while a seats–votes curve shows the likely seat share over all theoretically possible vote shares,[[4]](#footnote-4) only a small range around the typical statewide vote share are likely in practice.[[5]](#footnote-5) Hence, valid measures of bias measure it around the typical statewide vote share:

(5)

This requirement is analogous to the *principle of locality* in physics.[[6]](#footnote-6) Absent some theory and some empirical evidence to support it, there is no reason to believe that any metric that measures bias far away from the likely statewide vote share measures something related to bias close to it.

Hence, metrics that are not strictly functions of and don’t measure bias directly, on the one hand, and metrics that measure bias outside away from the typical statewide vote share don’t measure bias locally, on the other.

Finally, when you think about various and combinations, it becomes apparent that two things must be true for any valid measure of partisan bias:

* Axiom 1 – No misdiagnoses

1. If the Democratic vote share for a plan is greater than 0.5 and the corresponding seat share is *greater* than that vote share – i.e., and – a valid measure of bias will not indicate that the plan is biased in favor of Republicans.[[7]](#footnote-7)
2. Similarly, if the Republican vote share for a plan is greater than 0.5 and the corresponding seat share is greater than that vote share – i.e., and – a valid measure of bias will not indicate that the plan is biased in favor of Democrats.

Plans where a party will likely get more than half the seats when they receive more than half the votes may or may not be considered biased in favor of that party *depending on which measure of bias you choose.* More on that below.

* Axiom 2 – No failure to diagnose

1. If the Democratic vote share for a plan is greater than 0.5 but the corresponding seat share is *less* than that vote share – i.e., and – a valid measure of bias will indicate that the plan is biased in favor of Republicans.
2. Similarly, if the Republican vote share for a plan is greater than 0.5 but the corresponding seat share is less than that vote share – i.e., and – a valid measure of bias will indicate that the plan is biased in favor of Democrats.

Anti-majoritarian plans where one party gets more than half the votes but less than half the seats are a subset of this class of plans.

With a solid definition of bias in hand, we can now analyze various measures of bias that have been proposed.

2. Categories of Metrics

This section examines 10 candidate measures of bias.[[8]](#footnote-8) We classify them, evaluate them against the requirements for being a measure of partisan bias and conclude that just three actually measure bias in real life: (dis)proportionality (), the efficiency gap (), and gamma ().

## Measures of Symmetry

Three of the metrics measure the symmetry of Democratic & Republican district vote shares:

* Declination ()
* Lopsided outcomes (), and
* Mean–median ()

While these metrics are very useful diagnostic tools to help you understand when the voters favoring one party have been “packed” to achieve partisan advantage, they are not strictly functions of and so they don’t measure bias directly. Packing is a technique and frequently indicates partisan gerrymandering. These measures can be confounded though when states are unbalanced politically, e.g., one party typically wins 60 percent of the vote, because there are simply so many more people who tend to vote for the dominant party.

These are not direct measures of partisan bias.

## Non-Local Measures of Bias

Four other metrics do measure bias but fail the locality requirement:

* Seats bias ()
* Votes bias ()
* Geometric seats bias (), and
* Global symmetry ()

The first two measure bias at the point of symmetry not the typical statewide vote share .[[9]](#footnote-9) When the statewide vote share is close to the local region around will subsume and these measures may signal bias correctly. However, when a state is unbalanced politically, e.g., a 60/40 or 40/60 state, will be outside that local region around .

In contrast, *does* measure seats bias at the typical statewide vote share, but it uses the counterfactual at . Again, when the statewide vote share is close to this counterfactual is plausible and may signal bias correctly. But again, when the statewide vote share is unbalanced the counterfactual will fall outside the local region around it.

Global symmetry evaluates the asymmetry between the Democratic & Republican seats–votes curves over the entire range of theoretically possible vote shares, [0.0–1.0], almost all of which is outside the local region surrounding a typical statewide vote share.

Hence, even though these metrics may signal bias correctly when states are relatively balanced politically, they are non-local and unreliable when states are unbalanced politically.

## Measures of Local Bias

The last three metrics are – with one modification discussed in the next two sections – measures of bias as we have defined it:

* Proportional ()
* Efficiency gap (), and
* Gamma ()

They share the same underlying functional form:

(6)

where is an actual or idealized value of responsiveness:

* For proportionality, making the proportionality line where . On purely little ‘d’ democratic principles one might say that this the ideal seats–votes relationship.
* In contrast, the efficiency gap embeds an ideal responsiveness of 2—a two times winner’s bonus (). One can argue that this comports better with how single-member districts actually perform in practice.[[10]](#footnote-10)
* The gamma measure uses the responsiveness measured at the statewide vote share.

Since these measures are all evaluated at the statewide vote share and do not depend on counterfactuals outside the local range, I expect the signals from these metrics to match *a priori* expectations of bias even in states that are unbalanced politically.

## Measures of Responsiveness

In addition, we will make use of two measures of responsiveness:

* Overall responsiveness or winner’s bonus (), and
* Responsiveness ()

They are not measures of bias *per* se, but they inputs & context for the other measures.

3. Analysis of Select 2012 Congressional Plans

This section applies the candidate measures of partisan bias analyzed above to the selected congressional plans studied in (Nagle and Ramsay, 2021). In terms of their typical statewide two-party vote shares, four lean heavily Democratic (California, Illinois, Massachusetts, and Maryland), three lean heavily Republican (South Caroline, Tennessee, and Texas), and four are nearly balanced politically (Colorado, North Caroline, Ohio, and Pennsylvania).

The plans were drawn in 2011 and a composite of 2012 election results were used to evaluate them.[[11]](#footnote-11) The partisan profiles[[12]](#footnote-12) for these plans may be found in the Supplemental Material.[[13]](#footnote-13)

## 3.1. Expected Bias

Table 1 informs our expectations about the direction of bias (Republican or Democratic) and, to some extent, the degree of bias in each state’s plans.

Table

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Table 3–1

For each state and plan, there are two pieces of information:

* The typical statewide vote share (), and
* The body that controlled the process of drawing the plan – C = a commission | D = Democrats | R = Republicans | J = courts

When a state like Illinois leans heavily Democratic or a state like Tennessee leans heavily Republican, all else equal, because of the well-understood phenomenon of a “winner’s bonus” in single-member districting, one would expect the plans to be biased in favor of those parties even if the map-drawing body didn’t try to achieve partisan advantage. If one party or the other controlled the redistricting process, all else equal, one would expect that the resulting map would favor that party. In contrast, if a commission or court (re)drew a map, one would expect the plan to be relatively fair.[[14]](#footnote-14)

## 3.2. Observations

Table 2 evaluates the candidate metrics for the 2012 congressional plans for the sample states.[[15]](#footnote-15) The +/– *a priori* expectation of bias for each plan is captured in the last column, where plus indicates Republican bias and minus means Democratic bias.

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Table 3–2. Measures for 2012 Congressional Plans

These are seven sets of columns in the table:

* The 1st set shows the state, the statewide vote share (), and the corresponding likely seat share ( or simply ).
* The 2nd set shows the overall responsiveness or winner’s bonus for the plan () and the responsiveness at the typical statewide vote share ().
* The 3rd set shows the measures of symmetry: declination (), lopsided outcomes (), and mean–median difference ().
* The 4th set shows the non-local measures of bias: seats bias ( or ), votes bias ( or ), geometric seats bias (), and global symmetry ().
* The 5th set shows the candidate measures of bias: the deviation from proportionality (), the efficiency gap (), and gamma ().
* The last section shows three metrics that you can ignore for now. I will discuss them later.

The following sections compare these measurements to our *a priori* expectations of bias.[[16]](#footnote-16)

### 3.2.1. Measures of Symmetry

As hypothesized in the previous section, the three measures of symmetry don’t always match our *a priori* expectations of bias. For example, suggests Republican bias in the Illinois plan (positive sign) while our expectation is for Democratic bias (negative sign). This mismatch occurs in all four states that lean heavily Democratic as well as Texas that leaned heavily Republican in 2012.

Both declination and mean–median difference also send some unexpected +/– signals of bias, as illustrated by Illinois and Massachusetts, respectively.

### 3.2.2. Non-Local Measures of Bias

Similarly, as predicted in the previous section, the non-local measures of bias sometimes don’t always match our *a priori* expectations of bias. Again, you see this when the +/– signs of the metrics are the opposite of what we expect.

The Illinois plan illustrates how these measures can get confounded.

As the Fig. 3–1 shows, both the seats bias and votes bias measures suggest that the Illinois plan is biased in favor of Republicans despite our expectation that the plan favors Democrats. The reason is that the seats–votes curve passes the center point of symmetry[[17]](#footnote-17) down and to the right but crosses over the line of proportionality[[18]](#footnote-18) before reaching the statewide vote share of 60% where Democrats are likely to win almost 75% of the seats. This map clearly favors Democrats (negative sign), even though the seats and votes bias suggest the opposite (positive sign).

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Fig. 3–1. IL 2012 Congressional Plan

Even though the geometric seats bias () is evaluated at the statewide vote share, it depends on the counterfactual in which Democrats only get 40% of the statewide vote. In Illinois that Democratic vote share is very unlikely to happen. Hence, also incorrectly signals Republican bias for this plan.

Similarly, because global symmetry () evaluates asymmetry between the Democratic and Republican seats-votes curves over the entire range of vote shares instead of just around the statewide vote share it also erroneously signals Republican bias for the Illinois plan.

Seats bias and votes bias similarly send unexpected +/– signals of bias for the Massachusetts plan.

In a new twist, pun intended, the geometric seats bias () doesn’t send the wrong signal of bias for the Illinois plan. Fig. 3–2 shows why: while the Republican (red) seats–votes curve falls above the Democratic (blue) curve for most of the range between the point of symmetry and the point indicating Democratic bias, it twists under the Democratic curve before that likely actual point[[19]](#footnote-19) indicating Democratic bias.

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Fig. 3–2. MA 2012 Congressional Plan

### 3.2.3. Measures of Local Bias

In contrast to the non-local measures of bias, as hypothesized in section 2 the local measures of bias always match our *a priori* expectations of bias.

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Fig. 3–3. CO 2012 Congressional Plan

There are two interesting wrinkles to iron out here though:

* The for the Colorado plan – As described in Appendix A, the raw formula sends unexpected +/– signals of bias when the overall responsiveness or winner’s bonus () is between one and two inclusive. Because the winner’s bonus () for the Colorado plan is just 1.4, there is no bias according to . Hence, the highlighted value.
* The gamma () for the Illinois and Colorado plans – The raw 6.9% value (positive sign) for the Illinois plan indicates Republican bias even though our *a priori* expectation and all other indications for the plan are that it should be biased in favor of Democrats (negative sign). Again, as described in Appendix A, gamma uses the responsiveness () at the statewide vote share which is very high for this plan (3.1) and that multiplied by the vote share above half overwhelms the seat share above half. Like the post calculation heuristic, when the winner’s bonus for a plan is less than the inferred responsiveness but at least one (), there is an acceptable level of bias according to . The same is true for the Colorado plan.

These local measures of bias appropriately interpreted reliably match our *a priori* expectations of partisan bias for the sample 2012 congressional plans. This lends support for the definition of bias outlined in section 1.

4. Analysis of Corresponding 2020 Congressional Plans

This section analyzes the same metrics for the corresponding 2020 plans for the sample states using the 2016–2020 composite election results used in Dave’s Redistricting App, a popular free web-based redistricting tool.[[20]](#footnote-20) The plans are otherwise the same, except in North Carolina and Pennsylvania where courts redrew them mid-decade.

## 4.1. Observations

Table 3 shows the measurements for the 2020 plans.

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Table 4–1. Measures for 2020 Congressional Plans

The following sections compare these measurements to our expectations captured in the last column.[[21]](#footnote-21)

### 4.1.1. Measures of Symmetry

As predicted in section 2 and shown in section 3, these three measures of symmetry don’t always match our *a priori* expectations of bias. One or more have an unexpected +/– sign for the California, Illinois, Maryland, Massachusetts, Colorado, and Tennessee plans. seems particularly susceptible to indicating packing in states that are unbalanced politically, like Illinois.

### 4.1.2. Non-Local Measures of Bias

Similarly, as hypothesized in section 2 and shown in section 3, these non-local measures of bias also frequently don’t match our *a priori* expectations of bias. The reasons are easy to see in the seats–votes curves for Illinois, Massachusetts, and Texas.

The seats–votes curve for the Illinois plan (Fig. 4–1) passes the center point of symmetry down and to the right, so seats and votes bias both suggest that the plan is biased in favor of Republicans. According to Axiom 1 though, because the Democratic seat share (66.6%) is bigger than Democratic vote share (58.2%) a valid measure of bias can’t indicate *Republican* bias. Geometric seats bias () and global symmetry () also both send unexpected +/– signals for bias invalidating those metrics for violating Axiom 1.

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Fig. 4–1. IL 2020 Congressional Plan

The seats­­–votes curve for the Massachusetts plan (Fig. 4–2) has the same issues with respect to seats bias, votes bias, and global symmetry all suggesting Republican bias. However, because the Republican seats–votes curve (red) is *below* the Democratic seats–votes curve (blue) around the typical statewide vote share (61%), the geometric seats bias () has the expected +/– sign. Nonetheless, the counterfactual 39% statewide Democratic vote share is well outside the range of likely values.

The seats–votes curve for the Texas plan (Fig. 4–3) shows a state that leans Republican. Here the Democratic seats–votes curve (blue) passes the point of symmetry above and to the left suggesting that the plan is biased in favor or Democrats. The plan clearly favors Republicans though, as their seat share () is bigger than their vote share ().

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Fig. 4–2. MA 2020 Congressional Plan

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Fig. 4–3. TX 2020 Congressional Plan

### 4.1.3. Measures of Local Bias

As predicted and shown for the 2012 plans, when properly interpreted the local measures of bias always match our *a priori* expectations of bias:

* The efficiency gap () for the Colorado plan is positive suggesting Republican bias, even though *a priori* expectations and all other indications are that the plan has a slight Democratic bias. Here the winner’s bonus () is less than idealized responsiveness () of two that is embedded in the efficiency gap formula. Again, this indicates an acceptable level of bias using as one’s normative expectation of how votes should be translated into seats.
* Similarly, the gamma values for the Illinois and Texas plans have unexpected signs, because the winner’s bonus () in both cases are less than the relatively high responsiveness ().

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Fig. 4–4. CO 2020 Congressional Plan

These sample 2020 congressional plans add more support for the definition of bias in section 1.

5. Analysis of Hypothetical Plans

This section analyzes how the metrics we are investigating evaluate a carefully constructed set of hypothetical plans.

## 5.1. Hypothetical Plans

Warrington created a set of 12 hypothetical plans so he could study how well various metrics detected partisan gerrymandering (Warrington, 2019).[[22]](#footnote-22) Each plan has an associated partisan profile—a statewide vote share and district-by-district vote shares. The essence of each plan is illustrated by rank–votes graphs in Fig. 5–1. The details may be found in the Supplementary Material.

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Fig. 5–1. Rank–Votes Graphs for Warrington’s Hypothetical Plans

The statewide vote shares for these profiles and the corresponding estimated seat shares are shown in Table 5–1 below.[[23]](#footnote-23)

Table

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Table 5–1. Measures for Hypothetical Plans

Warrington evaluated the plans using first-past-the-post accounting, as opposed to the fractional seat probabilities that we use, so some of these scenarios may not report as crisply here. Nonetheless, because he designed them to illustrate various scenarios, most expectations of bias are simple and straightforward:

1. A: 1-proportionality – Designed to not be biased.
2. B: 2-proportionality – The seats–votes curve will be symmetric, and whether you think this is biased depends on your normative expectations of how votes should translate into seats.
3. C: 3-proportionality – Designed to be biased in favor of Democrats
4. D: Sweep – Even though the statewide vote share is only 64%, Democrats essentially win all the seats. Hence, this plan is designed to be biased in favor of Democrats.
5. E: Competitive – Even though statewide vote share is nearly even (52%) and there are several very competitive races, they all lean towards Democrats. Hence, this plan is designed to be biased in favor of Democrats.
6. F: Competitive even – Again, the statewide vote share is nearly even (51%) with several competitive districts. Here though none of them fall “in the ‘counterfactual window’ (i.e., between the majority party’s statewide support and 50%”[[24]](#footnote-24) and they all still lean Democratic. This plan is again designed to be biased in favor of Democrats.
7. G: Uncompetitive – This plan models an “uncompetitive election as might arise from a bipartisan gerrymander.”[[25]](#footnote-25) – the average winning margins for both parties are large. Because the statewide vote share slightly favors Democrats (52.3%), this plan is designed to be biased in favor of Democrats.
8. H: Very uncompetitive – This plan is like the previous example, except that the average winning margins are even more pronounced. This plan is designed to favor Democrats.
9. I: Cubic – This plan is designed to favor Democrats.
10. J: Anti-majoritarian – Here Democrats get less than half the votes but win more than half the seats, so this plan is designed to be biased in favor of Democrats.
11. K: Classic – This plan models a classic partisan gerrymander: Then statewide vote share is evenly split (50%), but “Republicans win a significant majority through having a number of narrow victories in contrast to their Democratic opponents whose few victories are overwhelming.”[[26]](#footnote-26) This plan is designed to be biased in favor of Republicans.
12. L: Inverted – This plan is a little tricky to intuit. It is somewhat complementary to the 2-proportionality example, except the Democratic & Republican vote shares are switched and more extreme. As above, whether you think this plan is biased depends on your normative expectations of how votes should translate into seats.

## 5.2. Observations

With those priors in place, we can now analyze how the metrics perform for the hypothetical plans:

* Both declination () and lopsided outcomes () send unexpected +/– signals of bias for the 2-proportionality and 3-proportionality scenarios.
* Mean–median difference () signals unexpectedly for the Competitive scenario and doesn’t signal any bias for the 3-proportionality and Sweep hypotheticals.
* The non-local measures of bias – seats bias (), votes bias (), geometric seats bias (), and global symmetry () – all send unexpected +/– signals of bias for the Competitive scenario.
* Moreover, as expected, geometric seats bias () also sends unexpected +/– signals everywhere the statewide vote share is very unbalanced.
* The efficiency gap () values needs to be interpreted in context as indicating no or acceptable bias in the 1-proportionality, 2-proportionality, and Inverted hypotheticals, because the winner’s bonus () is between one and two inclusive.
* Similarly, gamma () needs to be interpreted in context when the responsiveness at the statewide vote share () is large as in the Inverted scenario.

Because we expect to have to interpret and after computing the raw values, they reliably report the expected bias for Warrington’s hypothetical plans as does proportionality ().[[27]](#footnote-27)

6. Increasing Robustness

There are two ways that a measure of partisan bias can be made more robust. One is to incorporate the fact that seats are won in their entirety, and the other is to evaluate the measure over a local range that brackets the likely statewide vote share instead of a just one point. We illustrate this with the proportionality measure of bias ().

## 6.1. Using the Number of Seats Closest to Proportional

compares the likely *fractional* seat share () to the likely statewide vote share (). But you can’t win part of a seat: seats are won all or nothing. Hence, the measure in the tables above estimates disproportionality relative to the whole number of seats closest to proportional at the statewide vote share.[[28]](#footnote-28) This recognizes that the proportional ideal is better represented as a step function rather than the line .

As the tables above show, generally tracks , but there are times when they can diverge somewhat, e.g., both Colorado plans and the North Carolina and Pennsylvania 2020 plans.

## 6.2. Evaluating Over a Range Instead of at a Single Point

Proportionality (), the efficiency gap (), and gamma () are all evaluated at a single point: the likely statewide vote share (). But the statewide vote share is an *estimate* based on a composite of past elections, and it will ultimately vary somewhat for specific future elections. Hence, the metric in the tables above averages B% disproportionality over range of uncertainty that brackets the statewide vote share.[[29]](#footnote-29) In effect, estimates the area between the Democratic seats–votes curve (blue) and the number of seats closest to proportional step function for the local range of uncertainty (gray in the seats–votes curves).

As the tables above show, B% generally tracks , but again there are times when they can diverge somewhat. For example:

* CO 2012 Congressional – The responsiveness () at the statewide vote share is very high (3.8) and the number of seats closest to proportional changes from 3 to 4 in the zone of uncertainty, so it makes a difference where you evaluate (dis)proportionality.
* PA 2020 Congressional – Here again the responsiveness () is high (2.9) and the number of seats closest to proportional changes from 9 to 10 in the local range.

The number of seats closest to proportional changes in the local range for 8 of the 11 2012 plans: California, Illinois, Massachusetts, Colorado, North Carolina, Ohio, Pennsylvania, and Texas.

Evaluating it using whole seats and averaging samples over a range around the statewide vote share increases the robustness of the (dis)proportionality measure of partisan bias ().

Conclusion

Most popular measures of partisan bias either don’t measure it directly but instead measure a technique of partisan gerrymandering – declination (), lopsided outcomes (), and mean–median () – or aren’t reliable over a range of statewide vote shares that includes states that are politically unbalanced – seats bias (), votes bias (), geometric seats bias (), and global symmetry (). Gamma () measures partisan bias but isn’t useful when the estimated responsiveness () is high. In contrast, proportional () and efficiency gap () measure partisan bias directly and are reliable. The robustness of can be increased by using a step-function for the number of whole seats closest to proportional and averaging it over the local range of uncertainty around the likely statewide vote share.

References

*Citations need to be added throughout the manuscript.*

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Appendix A. Interpreting the Efficiency Gap & Gamma

After calculating the efficiency gap () and gamma () formulas, we apply a heuristic to interpret the results.

## A.1. Efficiency Gap

Since we use two-party Democratic vote shares, the formula for is:

(A–1)

Because we use two-party Democratic vote shares, negative values indicate Democratic bias, and positive values indicate Republican bias. The two in the formula idealizes a two-times winner’s bonus () when . In other words, the 2-proportional line is the only place where says there is no bias.

While signals clear Republican (+) and Democratic (–) bias correctly, it does not broadly indicate when a plan is *not* unduly biased in favor of one party or the other. Fig. A-1 below illustrates the issue.

The x-axis represents the Democratic statewide vote share () which ranges from [0.0, 1.0]. The y-axis represents the corresponding Democratic seat share (). Symmetric seats–votes curves pass the center point of symmetry, (0.5, 0.5). The dashed line is the line of proportionality with a slope of one. The dotted line has a slope of two and shows where . The vertical line represents a statewide vote share .

Diagram

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Fig. A-1. The Seats–Votes Space

The white circles illustrate three interesting regions in the upper right quadrant with analogous regions in the lower left quadrant. Two plans in our sample exemplify the most common points in this seats–votes space:

* CA 2020 Congressional ([map](https://davesredistricting.org/join/1f28107b-3981-46fc-9be5-6c3be848683d)) – As you can see in Table 4–1, this plan has a statewide vote share well over 0.5 and a winner’s bonus () over 2. That puts it in Region 1 in the upper right quadrant. The is negative (-4.6%) indicating bias in favor of Democrats.
* NC 2020 Congressional ([map](https://davesredistricting.org/join/87c865df-abda-40b4-aabb-59f9c388c3a0)) – This plan also has a winner’s bonus over 2 but the statewide vote is less than 0.5. This puts it in Region 1 in the lower left quadrant. The is positive (7.5%) indicating bias in favor of Republicans.

Sometimes the point for a plan falls in Region 3 in the upper right quadrant where is positive which indicates Republican bias:

* PA 2020 Congressional ([map](https://davesredistricting.org/join/3d91ae72-035c-49d8-920b-d44012d3ea2d)) – Here Democrats get more than half the votes but a disproportionally *smaller* seat share (). The winner’s bonus is less than one.

There can be similar examples in Region 3 in the lower left quadrant. We label these regions sub-proportional.[[30]](#footnote-30) The lower right and upper left quadrants are parts of Region 3 where plans are anti-majoritarian: one party gets more than half the votes but wins less than half the seats.

Occasionally the vote share and seat share for a plan will fall into Region 2 in the upper right quadrant. For example:

* CO 2020 Congressional ([map](https://davesredistricting.org/join/22b4dc91-5c29-4d11-a408-108feda65390)) – In this case, the base formula returns a positive number indicating Republican bias, even though the Democratic seat share is bigger than the vote share . At 1.95, the winner’s bonus () falls between one (proportionality) and two () and the is 0.2%.

In situations like this when the winner’s bonus () is between one and two, we interpret the plans as having an acceptable level of bias according to . The same holds for the analogous Region 2 in the lower left quadrant.

We highlight the Colorado in Table 3 to indicate this.

## Gamma

If slightly more complicated, the analogous issue exists for Nagle’s gamma () measure. Instead of the two times winner’s bonus idealized in , uses the responsiveness () estimated at statewide vote share:

(A–2)

The same three scenarios exist for :

* When the winner’s bonus is greater than the responsiveness () or the winner’s bonus is less than one (), negative values indicate Democratic bias and positive values indicate Republican bias, but
* When the winner’s bonus is less than the responsiveness but at least one (), there is an acceptable level of bias according to .

Two plans in Table 4–1 have gamma values that fall into this last category:

* IL 2020 Congressional ([map](https://davesredistricting.org/join/1f28107b-3981-46fc-9be5-6c3be848683d)) –
* TX 2020 Congressional ([map](https://davesredistricting.org/join/19b1b774-7706-485b-a21a-896bbcbddbba)) –

In both cases, the responsiveness at the statewide vote share is quite high: 2.9 and 3.1, respectively.

As with the Region 2 values above, we highlight these values in Tables 3–2, 4–1, and 5–1 to remind us to ignore the raw measurement.

Appendix B. Classifying Seats–Votes Curves

Images of the seats–votes curves for all the plans we have studied may be found in the Supplementary Material.

Seats–votes curves can be classified on where the statewide vote share & likely seat share point falls. The seats–votes space may be divided into three pairs of regions:

* Super-proportional
* Sub-proportional, and
* Anti-majoritarian

Each is discussed below.

## B.1. Super-proportional

When and , Democrats get more than half the votes and their seat share is bigger than their vote share. Conversely, when and , Republicans get more than half the votes and their seat share is bigger than their vote share.

Curves that fall into these regions may be further divided into curves that do or don’t cross the line of proportionality between the point of symmetry and the likely actual point and curves that are symmetric.

Curves that don’t cross over the line of proportionality give unambiguous signals about bias:

* Hypotheticals F: Competitive even, G: Uncompetitive, H: Very uncompetitive, K: Classic,[[31]](#footnote-31) and L: Inverted
* Both the 2012 and 2020 plans for CA, MD, SC, TN, and TX
* The NC 2020 map which was re-drawn by the courts, and
* The OH 2020 map

Curves that do cross over the line of proportionality give mixed signals about bias. These curves indicate one thing around the point of symmetry and another around the likely actual point:

* Both the 2012 and 2020 plans for CO, IL, and MA

Curves that pass through the point of symmetry are symmetric:

* The hypothetical plans A: 1-proportionality, B: 2-proportionality, C: 3-proportionality, D: Sweep, E: Competitive, and I: Cubic

Of course, a vote share of exactly 50% is unlikely in real-life.

## B.2. Sub-proportional

When and but , Democrats get more than half the votes and win more than half the seats but their seat share is smaller than their vote share. The converse happens for Republicans, when and but . One plan in our sample falls into this category:

* The PA 2020 plan

This was re-drawn by the PA Supreme Court.

## B.3. Anti-majoritarian

When but for a plan, Democrats get more than half the votes but less than half the seats, an anti-majoritarian result. If and , Republicans similarly get more than half the votes but Democrats win more than half the seats.

* The hypothetical plan J: Anti-majoritarian illustrates this, and
* The NC, OH, and PA 2012 plans are real-life examples

The 2012 NC and PA plans were later re-drawn by courts. The NC 2020 plan became super-proportional while the PA 2020 plan became sub-proportional. Over the decade, the statewide vote share in OH shifted enough (51% to 46%) to make the OH 2020 super-proportional.

[end]

1. We use two-party Democratic vote shares by convention. Two-party Replication vote shares are simply . [↑](#footnote-ref-1)
2. We use John Nagle’s base methodology of fractional seat probabilities described in <https://lipid.phys.cmu.edu/nagle/Technical/FractionalSeats2.pdf> and a seats–votes curve inferred using proportional shift described in <https://lipid.phys.cmu.edu/nagle/Technical/2019-04-19%20-%20Measuring%20Redistricting%20Bias%20&%20Responsiveness.pdf>. [↑](#footnote-ref-2)
3. John Nagle has called this <V> in the past. [↑](#footnote-ref-3)
4. Because statewide vote shares tend to not fall much outside the range [0.4, 0.6], we only infer the points of the seats–votes curve for the range [0.25, 0.75]. [↑](#footnote-ref-4)
5. As will be described below, we use a 5% range that brackets the statewide vote share, because the average uncertainty for the seats–votes curves in (Nagle and Ramsay, 2021) was roughly 2%. [↑](#footnote-ref-5)
6. <https://en.wikipedia.org/wiki/Principle_of_locality> [↑](#footnote-ref-6)
7. To make formulas easier to write, we represent percentages as [0–1] fractions in the body text. Except as noted, in tables and figures, we show them as percentages. [↑](#footnote-ref-7)
8. They are defined in (Nagle and Ramsay, 2021) and shown in Table 1 there. Here they are computed in Tables 2–4 below. [↑](#footnote-ref-8)
9. Since votes bias measures an incremental *vote* share, it doesn’t strictly meet our definition of a measure of bias, but one can think of it as a complement to seats bias. [↑](#footnote-ref-9)
10. (Stephanopoulos & McGhee, 2018): “a fourth parameter is *empirical correspondence*. That is, the electoral ideal implied by a metric should not be too different from the American historical norm. Otherwise the measure would imply that most American plans have been gerrymanders—and its adoption would be so disruptive as to be infeasible.” [↑](#footnote-ref-10)
11. We call them “2012” plans hereafter. [↑](#footnote-ref-11)
12. A partisan profile consists of a state vote share and district-by-district vote shares, using Democratic two-party votes. [↑](#footnote-ref-12)
13. Supplementary Material is available in this GitHub repository: <https://github.com/dra2020/bias-irl>. [↑](#footnote-ref-13)
14. Because both states are very unbalanced politically in favor of Democrats and because Democrats drew both of the maps, our *a priori* expectation is that the Illinois and Massachusetts plans are biased in favor of Democrats. McDonald et al. (2018, 323) concluded that the Illinois 2012 map was not a Democratic gerrymander and acknowledged that political geography in Illinois favored Republicans, but they didn’t claim that the map favored Republicans. McGann et al. (2016, 105) concluded that the Illinois map was not strongly biased in favor of Democrats and also acknowledged that the state’s political geography favored Republicans, but they didn’t claim that the map favored Republicans. McDonald et al's conclusions with respect to the 2012 Massachusetts map are similar, while McGann et al concluded that the Massachusetts plan was unbiased. [↑](#footnote-ref-14)
15. This is a slightly revised version of Table 1 in (Nagle and Ramsay, 2021). The columns have been reordered to match the categories of metrics in the previous section and a few metrics have been added as has a +/– *a priori* expectation of bias. [↑](#footnote-ref-15)
16. The metrics were computed using DRA’s analytics engine: <https://github.com/dra2020/dra-analytics>. Seats­–votes curves were plotted using the associated standalone application: <https://github.com/dra2020/dra-analytics-app>. [↑](#footnote-ref-16)
17. The **point of symmetry** is the point in the center of the seats–votes space. All symmetric seats–votes curves pass through this point. [↑](#footnote-ref-17)
18. The **line of proportionality** is line through the point of symmetry where . [↑](#footnote-ref-18)
19. The **likely actual point**is the point that represents the likely statewide Democratic two-party vote share and the likely corresponding seat share. [↑](#footnote-ref-19)
20. These can be found in the Official Maps collection of Dave’s Redistricting App: <https://davesredistricting.org/>. [↑](#footnote-ref-20)
21. The metrics were computed using the *Advanced* tab in DRA. [↑](#footnote-ref-21)
22. Note: His notion of “fair” is rooted in partisan symmetry. [↑](#footnote-ref-22)
23. We used whole seats instead of fractional seats for the column, to make the values simpler. [↑](#footnote-ref-23)
24. Warrington, 2019. [↑](#footnote-ref-24)
25. Warrington, 2019. [↑](#footnote-ref-25)
26. Warrington, 2019. [↑](#footnote-ref-26)
27. Because we use fractional seat probabilities as opposed to the first-past-the-post accounting that Warrington used, the 1-proportionality case doesn’t evaluate as purely proportional. [↑](#footnote-ref-27)
28. This is the measure of (dis)proportionality used in DRA in the *Proportionality* section of the *Analyze* tab. [↑](#footnote-ref-28)
29. Remember, we define “local” to be the 5% range that brackets the statewide vote share, i.e., +/– 2.5%. [↑](#footnote-ref-29)
30. See Appendix B. “Classifying Seats–Votes Curves.” [↑](#footnote-ref-30)
31. The statewide vote share for this hypothetical example is exactly 50%. To classify it per Warrington’s intent as a classic gerrymander, I assume the Democratic vote share was just slightly less than that. [↑](#footnote-ref-31)